Compound Matrices in Differential Equations

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2.1 Compound Matrices and Differential Equations

2.1.1 Minors, Cofactor and Adjugates

Definition 1. $a^{s_1\dots s_p}_{r_1\dots r_p} = \det[a^{s_i}_{r_i}]$, $1 \le i, j \le p$, *is the minor of* A *determined by the rows* $r_1, ..., r_p$ *and the columns* $s_1, ..., s_p$.

Definition 2. *When* $p < m = n$, $A^{s_1,\dots,s_p}_{r_1,\dots r_p}$ *denotes the cofactor of* $a^{s_1\dots s_p}_{r_1\dots r_p} = \det[a^{s_i}_{r_i}]$.

When $A = \begin{pmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_3 & a^2_3 & a^3_3 \end{pmatrix}$, then we have $A^2_3 = -a^{13}_{12} = -\begin{vmatrix} a^1_1 & a^2_1 \\ a^1_3 & a^2_3 \end{vmatrix}$

Definition 3. *The cofactor matrix of a square matrix* A *is*

$$\text{cof } A = [A^j_i],\ i, j = 1, ..., n$$

and the adjugate (or classical adjoint) matrix of A is

$$\text{adj}A = (\text{cof}A)^T$$

2.1.2 Multiplicative Compounds

For any $m \times n$ matrix A and $1 \le k \le \min m, n$, the k-th multiplicative compound $A^{(k)}$ of A is the $\binom{m}{k} \times \binom{n}{k}$ -dimensional matrix defined as follows.

Definition 4. *If* $1 \le r \le \binom{m}{k}$ *and* $1 \le s \le \binom{n}{k}$*, then the entry in the r-th row and the s-th column of* $A^{(k)}$ *is* $a_{r_1 \ldots r_k}^{s_1 \ldots s_k}$*, where* $(r) = (r_1, ..., r_k)$ *is the r-th member of the lexicographic ordering of the integers* $1 \le r_1 < r_2 < ... < r_k \le m$ *and* $(s) = (s_1, ..., s_k)$ *is the s-th member in the lexicographic ordering of all k-tuples of the integers* $1 \le s_1 < s_2 < ... < s_k \le n$*.*

Thus, if $A = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ \vdots & \vdots \\ a_m^1 & a_m^2 \end{bmatrix}$, then we have $A^{(2)} = \begin{bmatrix} a_{12}^{12} \\ a_{13}^{12} \\ \vdots \\ a_{m-1,m}^{12} \end{bmatrix}$, which is a $\binom{m}{2} \times 1$ matrix.

Theorem 1. *If* $AB = C$*, then* $A^{(k)}B^{(k)} = C^{(k)}$*, where* A, B *are* $n \times p$*,* $p \times n$ *matrices respectively.*

Theorem 2. *The eigenvalues of* $A^{(k)}$ *are* $\lambda_{s_1}\lambda_{s_2}...\lambda_{s_k}$*,* $1 \le s_1 < s_2 < ... < s_k \le n$*, where* $\lambda_1, ..., \lambda_n$ *are the eigenvalues of A.*

2.1.3 Additive Compounds

Let A be a $n \times n$ matrix and let $1 \le k \le n$. Then the *k-th additive compound* $A^{[k]}$ of A is a $\binom{n}{k} \times \binom{n}{k}$ matrix defined as follows.

Definition 5.

$$A^{[k]} = \frac{d}{dt}(I + tA)^{(k)}|_{t=0} = \lim_{h \to 0} h^{-1}[(I + hA)^{(k)} - I^{(k)}] \tag{1}$$

It follows that the entry b_r^s in $B = A^{[k]}$ is:

$$b_r^s = \begin{cases} a_{r_1}^{r_1} + \cdots + a_{r_k}^{r_k}, & \text{if}(r) = (s) \\ (-1)^{i+j} a_{r_i}^{s_j}, & \text{if exactly one entry } r_i \text{ in } (r) \text{ does not occur in } (s) \text{ and } s_j \\ & \text{does not occur in } (r) \\ 0, & \text{if } (r) \text{ differs from } (s) \text{ in two or more entries} \end{cases} \tag{2}$$

In the special cases $k = 1, k = n$, we find

$$A^{[1]} = A, \quad A^{[n]} = \text{Tr}A$$

Properties 1. *The term additive compound arises since*

$$(A + B)^{[k]} = A^{[k]} + B^{[k]} \tag{3}$$

Theorem 3. *The eigenvalues of* $A^{[k]}$ *are* $\lambda_{s_1} + \lambda_{s_2} + ... + \lambda_{s_k}$, $1 \leq s_1 < s_2 < ... < s_k \leq n$, *where* $\lambda_1, ..., \lambda_n$ *are the eigenvalues of* A.

Theorem 4. *For* $n \times n$ *matrix* A, *the equation holds:*

$$(\exp(A))^{(k)} = \exp(A^{[k]}) \tag{4}$$

2.2 Linear Operators

In this section we consider multiplicative and additive compounds from the perspective of linear operators between linear spaces.

Definition 6. *Suppose that* $\mathbb{X}$ *is an n-dimentional linear space. Then* $(\wedge\mathbb{X})^k$ *is the k-th wedge sapce whose bases are* $e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$, $1 \leq i_1 < i_2 < ... < i_k \leq n$. *And the entries of* $v^1 \wedge v^2 \wedge ... \wedge v^k$ *at* $e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$ *are* $\det(v^s_{i_j})$, $1 \leq j, s \leq k$

Proposition 1. *Suppose* A *is an* $n \times n$ *matrix and viewed as an operator on the linear space* $\mathbb{X}$, *then* $A^{(k)}$ *and* $A^{[k]}$ *are matrices viewed as operators on* $(\wedge\mathbb{X})^k$, *satisfying:*

$$A^{(k)}(v^1 \wedge v^2 \wedge ... \wedge v^k) = Av^1 \wedge Av^2 \wedge ... \wedge Av^k \tag{5}$$

$$A^{[k]}(v^1 \wedge v^2 \wedge ... \wedge v^k) = \sum_{s=1}^{k} v^1 \wedge ... \wedge Av^s \wedge ... \wedge v^k \tag{6}$$

We can prove all of the previous theorems and properties with Properties**??**.

2.3 Nonhomogeneous differential equations

To consider about the generalized problem, we need to firstly review the original form. We have homogeneous equation

$$\dot{x} = A(t)x \tag{7}$$

And the nonhomogeneous equation

$$\dot{x} = A(t)x + f(x) \tag{8}$$

Definition 7. *Let* $X(t)$ *be a fundamental matrix for (7) and let* $|\cdot|$ *denote any matrix norm. We may assume without loss of generality that the norm is induced by a vector norm. The equation is said to be*

1. *Stable if there is a constant* K *such that* $|X(t)| \leq K$, $0 \leq t < \infty$.
2. *Asymptotically stable if* $|X(t)| \to 0$, *as* $t \to \infty$
3. *Uniformly stable if there exists a constant* K *that* $|X(t)X^{-1}(t_0)| \leq K$, $0 \leq t_0 \leq t < \infty$.
4. *Uniformly asymptotically stable if there exist constants* $K, \alpha > 0$ *such that* $|X(t)X^{-1}(t_0)| \leq Ke^{-\alpha(t-t_0)}$, $0 \leq t_0 \leq t < \infty$

To further get some properties, we need to give some restrictions on those solutions.

$$\begin{aligned} &(i) \quad \limsup_{t\to\infty} |y(t)| < \infty, \\ &(ii) \quad \liminf_{t\to\infty} |y(t)| = 0 \Rightarrow \lim_{t\to\infty} y(t) = 0. \end{aligned} \tag{L}$$

Proposition 2. *Suppose that the homogeneous equation (7) is uniformly stable and* $f \in L_1([0,\infty)])$. *Then the solution space of the nonhomogeneous equation (8) satisfies condition (L).*

Proposition 3. *Suppose that the homogeneous equation (7) is uniformly stable and asymptotically stable and that* $f \in L_1([0,\infty))$. *Then all solutions* $x = x(t)$ *of the nonhomogeneous equation (8) satisfy* $\lim_{t\to\infty} x(t) = 0$.

Proposition 4. (Muldowney, 1990). *Let* $\mathcal{X}$ *be a linear space of functions* $x : [0,\infty) \to \mathbb{R}^n$ *that satisfies* (L). *Then*

$$\text{codim } \mathcal{X}_0 < k \Leftrightarrow \mathcal{X}_0^{(k)} = \mathcal{X}^{(k)}.$$

Here $\mathcal{X}^{(k)}$ *denotes the k-th exterior power of* $\mathcal{X}$, $1 \le k \le n$, *which is defined by*

$$\mathcal{X}^{(k)} = span\{x^1 \wedge \cdots \wedge x^k : \; x^i \in \mathcal{X}\}.$$

and $\mathcal{X}_0$ *and* $\mathcal{X}_0^{(k)}$ *denote subspaces of* $\mathcal{X}$ *and* $\mathcal{X}^{(k)}$, *respectively, defined by*

$$\mathcal{X}_0 = \{x \in \mathcal{X} : \lim_{t\to\infty} x(t) = 0\},$$
$$\mathcal{X}_0^{(k)} = \{w \in \mathcal{X}^{(k)} : \lim_{t\to\infty} w(t) = 0\}.$$

Using the three previous propositions, we can state one of the main results:

Theorem 5. *Let* $x_i(t)$ *be a solution of*

$$x' = A(t)x + f_i(t),$$

where $f_i \in L_1[0,\infty)$, $i = 1, ..., k$. *Suppose that the homogeneous equation*

$$y' = A(t)y$$

is uniformly stable and that the k-th compound equation

$$z' = A^{[k]}(t)z$$

is uniformly stable and asymptotically stable. Then there exist constants $c_1, ..., c_k$, *not all zero,such that*

$$\lim_{t\to\infty} (c_1x_1(t) + \cdots + c_kx_k(t)) = 0,$$

3 Description of the general form

Consider the evolution in time of objects like

$$z(t) = x(t) \wedge y(t). \tag{9}$$

where x, y are solutions of homogeneous differential equations $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$, respectively.

We have shown that $\dot{z} = A^{[2]}z$ if $B(t) = A(t)$.Now we try to find out if similar properties hold for general z.

3.1 Nonexistence of the coefficient matrix

Firstly we want to know if such a matrix M exists, such that $z' = M(t)z$. The solutions x and y to two homogeneous equations can be viewed to be unralated if there is no restrictions on $A(t), B(t)$.

Proposition 5. *If the solutions $x(t)$ and $y(t)$ are independent, i.e. the entries $x_1(t), x_2(t), ..., x_n(t)$, $y_1(t), y_2(t), ..., y_n(t)$ are viewed as bases of vector spaces, then the matrix $M(t)$ doesn't exist.*

Proof. To prove the proposition, we simply expand z and its derivative z'. It is easily seen that

$$z' = x'(t) \wedge y(t) + x(t) \wedge y'(t) = A(t)x(t) \wedge y(t) + x(t) \wedge B(t)y(t).$$

Suppose the basis of the space $\mathbb{R}$ is $e^1, ..., e^n$ and $e_1, ..., e_n$ be the coordinate operator. Then the 2-th basis is $e^i \wedge e^j$, $1 \leq i < j \leq n$.

we compare the coordinate at $e^i \wedge e^j$.

For $z = x(t) \wedge y(t)$, we have

$$\begin{aligned}\langle e_i \wedge e_j, x(t) \wedge y(t)\rangle &= \det \begin{pmatrix} e_i(x(t)) & e_i(y(t)) \\ e_j(x(t)) & e_j(y(t)) \end{pmatrix} \\ &= x^i(t)y^j(t) - y^i(t)x^j(t).\end{aligned}$$

And for $z' = A(t)x(t) \wedge y(t) + x(t) \wedge B(t)y(t)$, we have

$$\begin{aligned}\langle e_i \wedge e_j, A(t)x(t) \wedge y(t)\rangle &= \det \begin{pmatrix} e_i(A(t)x(t)) & e_i(y(t)) \\ e_j(A(t)x(t)) & e_j(y(t)) \end{pmatrix} \\ &= \Big(\sum_{k=1}^{n} a_{ik}(t)x^k(t)\Big)y^j(t) - \Big(\sum_{k=1}^{n} a_{jk}(t)x^k(t)\Big)y^i(t).\end{aligned}$$

$$\begin{aligned}\langle e_i \wedge e_j, x(t) \wedge B(t)y(t)\rangle &= \det \begin{pmatrix} e_i(x(t)) & e_i(B(t)y(t)) \\ e_j(x(t)) & e_j(B(t)y(t)) \end{pmatrix} \\ &= \Big(\sum_{k=1}^{n} b_{jk}(t)y^k(t)\Big)x^i(t) - \Big(\sum_{k=1}^{n} b_{ik}(t)y^k(t)\Big)x^j(t).\end{aligned}$$

Observe that the coordinate of z has a form of commutator. Compare it with z' we can easily see that if such a matrix $M(t)$ exists and $A(t), B(t)$ are not degenerate, we must force $a_{ik} = b_{ik}$, which implies that $A(t) = B(t)$, $z' = A^{[2]}z$, the former conclusion. □

3.2 Special cases

In the previous section, we have shown that $M(t)$ does not exist for generic $A(t)$ and $B(t)$, because $x(t)$ and $y(t)$ are free. Now we try to examine z under some special conditions.

Example 1. *The easily condition is diagonalization, under which we can direct solve the solutions $x(t), y(t)$. Assume $A(t), B(t)$ are similar with a diagonal matrix $\Lambda(t)$ by constant matrices P, Q, respectively. i.e.*

$$x'(t) = P^{-1}\Lambda(t)P\ x(t)$$
$$y'(t) = Q^{-1}\Lambda(t)Q\ y(t)$$

Consider the equation $w'(t) = \Lambda(t)w(t)$, we can easily get its fundamental matrix

$$W(t) = \begin{pmatrix} w_1(t) & & & \\ & w_2(t) & & \\ & & \ddots & \\ & & & w_n(t) \end{pmatrix}$$

where $w_i = \exp(\int \lambda_i(s)ds)$. Then the fundamental matrix for x, y are $P^{-1}W(t)$ and $Q^{-1}W(t)$.

Example 2. *When $A(t), B(t)$ are similar by a nonsingular constant matrix P, i.e. $B(t) = PA(t)P^{-1}$. If $X(t)$ is a fundamental matrix for $x' = A(t)x$ then $PX(t)$ would be a fundamental matrix for $y' = B(t)y$. Suppose that $x^i(t)$ and $x^j(t)$ are solutions to $x' = A(t)x$. We have*

$$\begin{aligned} z(t) &= x^i(t) \wedge Px^j(t), \\ z'(t) &= Ax^i(t) \wedge Px^j(t) + x^i(t) \wedge PAx^j(t). \end{aligned}$$

the evolution of $z(t)$ can be calculated directly if we have properties of $X(t)$. But under the assumption that $A(t), B(t)$ are similar, the nonexistence of $M(t)$ still holds.

3.3 Local problem

We have seen that the holistic system doesn't hehave well when we focus on the matrix $A(t)$ and $B(t)$. So we begin with solutions now.

We assume that all solutions $x(t)$ to $x' = A(t)x$ satisfies condition (L), and $y(t)$ as well.

Proposition 6. *If $\forall t \in [0, +\infty)$, $|A(t) - B(t)| \neq 0$, then the solutions of $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ are linearly independent.*

Proof. We only need to prove the solution space W_1 of $\dot{x} = A(t)x(t)$ and solution space W_2 of $\dot{y} = B(t)y$ satisfy $W_1 \cap W_2 = \{0\}$. Let the fundamental matrix of $\dot{x} = A(t)x$ be $X(t)$, we will prove that for every $c \in \mathbb{R}^n/\{0\}$, $X(t)c$ is not a solution to $\dot{y} = B(t)y$.

Suppose $X(t)c$ satisfies $\dot{X(t)}c = B(t)X(t)c$, then $(\dot{X(t)} - B(t)X(t))c = 0$. As $X(t)$ satisfies $\dot{X} = A(t)X(t)$, implying $(A(t) - B(t))X(t)c = 0$. Since $X(t)$ is non-singular,

$$|(A(t) - B(t))X(t)| = |(A(t) - B(t)||X(t)| \neq 0.$$

So the only solution is zero, contradicting with $c \in \mathbb{R}^n/\{0\}$. Therefore $\forall c \in \mathbb{R}^n/\{0\}$, $X(t)c$ is not a solution to the equation $\dot{y} = B(t)y$. i.e. $W_1 \cap W_2 = \{0\}$ ☐

Theorem 6. *Suppose that $x(t)$ is a solution of $\dot{x} = A(t)x$ and that $y(t)$ is a solution of $\dot{y} = B(t)y$, where $|A(t) - B(t)| \neq 0$, $\forall t \in [0, \infty)$. Let $z(t) = x(t) \wedge y(t)$ be the wedge product of them. Then there exists an $n \times n$ matrix $C(t)$, which satisfies $\dot{z}(t) = C^{[2]}z(t)$.*

Proof. Take $k-1$ solutions in the solution system of $\dot{x}(t) = A(t)x(t)$ that are linearly independent with $x(t)$ and denote them as $x^2(t), ..., x^k(t)$. And take another $n-k-1$ vectors in the solution system of $\dot{y}(t) = B(t)y(t)$ that are linearly indepent with y(t), and denote them as $y^{k+2}(t), ..., y^n(t)$. Let

$$W(t) = [x(t), x^2(t), ..., x^k(t), y(t), y^{k+2}(t), ..., y^n(t)]$$

By Proposition6, we know that $W(t)$ is non-singular.

Let $C(t) = \dot{W}(t)W^{-1}(t)$, which is $\dot{W} = C(t)W(t)$, we know that $x(t), y(t)$ are both solutions to this homogeneous equation. By the former propositions, we get $\dot{z}(t) = C^{[2]}(t)z(t)$. $\square$

Proposition 7. *Suppose* $|A(t) - B(t)| \neq 0$*, C(t) is an arbitrary matrix, define*

$$r_A \xlongequal{def} rank(A(t) - C(t))$$
$$r_B \xlongequal{def} rank(B(t) - C(t))$$

are constants for all $t \in [0, \infty)$*.*

then the equation $\dot{z} = C^{[2]}z$ *have* $(n - r_A)(n - r_B)$ *solutions in the form* $x(t) \wedge y(t)$*.*

Proof. There is an underlying condition

$$r_A + r_B = rank(A(t) - B(t)) + rank(B(t) - C(t)) \geq rank(A(t) - B(t)) = n$$

Back to our proof, according to the conditions, we can find $n - r_A$ solutions $x^i(t)$, $i = 1, ..., n - r_A$ and $n - r_B$ solutions $y^j(t)$, $j = 1, ..., n - r_B$ such that $(A(t) - C(t))x^i(t) \equiv 0$ and $(B(t) - C(t))y^j(t) \equiv 0$, $(n - r_A) + (n - r_B) \leq n$.

Meanwhile, the solutions $x^1(t), ..., x^{n-r_A}, y^1, ..., y^{n-r_B}$ are linearly independent by Proposition6. So it can be expanded to a fundamental matrix $W(t)$ of $\dot{z} = C(t)z$. Then we have $x^i(t) \wedge y^j(t)$ $(1 \leq i \leq n - r_A,\ 1 \leq j \leq n - r_B)$ are $(n - r_A)(n - r_B)$ solutions. $\square$

Now we turn to study on holistic system of some selected solutions. Recall that we have assumed that all solutions to homogeneous equations $A(t), B(t)$ satisfy condition (L).

Lemma 1. *Suppose* $x(t), y(t)$ *are solutions to* $\dot{x} = A(t)x$ *and* $\dot{x} = B(t)x$*, respectively. And* $x(t)$ *and* $y(t)$ *are linearly independent. And* $\exists\ C(t)$ *such that* $x(t), y(t)$ *are solutions to* $\dot{z}(t) = C(t)z$ *and all solutions to* $\dot{w}(t) = C(t)w$ *satisfies* (L)*. if* $\lim_{t\to+\infty} x(t) \wedge y(t) = 0$*, there are three possible conditions*

(1) $\lim_{t\to\infty} x(t) = 0$.

(2) $\lim_{t\to\infty} y(t) = 0$.

(3) $\exists\ c_1, c_2 \in \mathbb{R}/\{0\}$ *such that* $\lim_{t\to\infty} c_1x(t) + c_2y(t) = 0$.

Proof. (1) and (2) are obvious, for $x(t)$ and $y(t)$ are bounded because of (L).

Now we assume that both $x(t)$ and $y(t)$ have a distance from 0. According to (L), there is a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that both $x(t_k), y(t_k)$ have limits when $k\to\infty$, denote as $x^0, y^0 \neq 0$.

By the hypothesis, $\lim_{k\to\infty} x(t_k)\wedge y(t_k) = x^0\wedge y^0 = 0$, then there exists $c_1, c_2 \in \mathbb{R}/\{0\}$ such that $c_1x^0+c_2y^0 = 0$. We have $\lim_{k\to\infty} c_1x(t_k)+c_2y(t_k) = 0$. Since $c_1x(t_k)+c_2y(t_k)$ is a solution of $\dot{w} = C(t)w$ and it gives $\lim_{t\to\infty} c_1x(t) + c_2y(t) = 0$ by condition (L). □

Now we begin to study the condition when $\lim_{t\to\infty} x(t)\wedge y(t) = 0$ for all solutions x and some y. Denote them as $x^1, ..., x^n$ and $y^1, ..., y^s$, $(1 \le s \le n)$.

The condition in Proposition6 is strong, without which we can also get some properties.

Example 3. *If $y^1, ..., y^s$ are the linear combination of $x^{n-s+1}, ..., x^n$,* i.e.

$$[y^1, y^2, ..., y^s] = [x^{n-s+1}, ..., x^n]P,$$

where P is an $s\times s$ non-singular matrix, then the matrix is

$$\begin{aligned} W(t) &= [x^1, ..., x^{n-s}, y^1, ..., y^s] \\ &= [x^1, ..., x^{n-s}, x^n - s + 1, ..., x^n] \begin{pmatrix} I_{n-s} & \\ & P \end{pmatrix} \\ &= X(t) \begin{pmatrix} I_{n-s} & \\ & P \end{pmatrix} \end{aligned}$$

where $X(t)$ is the fundamental matrix of $\dot{x} = A(t)x$. Let $C(t) = \dot{W(t)}W^{-1}(t)$ and we have

$$\begin{aligned} C(t) = \dot{W}(t)W^{-1}(t) &= \dot{X}(t) \begin{pmatrix} I_{n-s} & \\ & P \end{pmatrix} \left(X(t) \begin{pmatrix} I_{n-s} & \\ & P \end{pmatrix} \right)^{-1} \\ &= \dot{X}(t)(X^{-1}(t) = A(t) \end{aligned}$$

Set $s_0 = \min\{n-s, s\}$ we can see that $\dot{z}(t) = C^{[s_0+1]}z(t)$ is asymptotically stable. So is $\dot{x}(t) = A^{[s_0+1]}x(t)$

Remark 1. *Observe that if all $x^i(t)$ converge to 0 as $t\to\infty$, we have $s = n$, And the equation $\dot{x}(t) = A(t)x(t)$ is asymptotically stable.*

Remark 2. *If a solution $y(t)$ of $\dot{y}(t) = B(t)y(t)$ converges to 0 as $t\to\infty$, then $x^i(t)\wedge y(t) \to 0$ as $t\to\infty$ for all $i = 1, 2, ..., n$, which means $y(t) \in span[y^1(t), ..., y^s(t)]$.*

So suppose that $x^1, ..., x^{n_1}$ $(0 \le n_1 \le n)$ converge to 0, and $y^1, ..., y^{s_1}$ $(0 \le s_1 \le s)$ converge to 0. If there are solutions $x(t), y(t)$ that doesn't converge to 0. i.e. $n_1 < n, s_1 < s$, we can find $c_1, c_2 \in \mathbb{R}/\{0\}$ such that $\lim_{t\to\infty} c_1x(t)+c_2y(t) = 0$ by Lemma1.

We abstract the properties into a proposition, but we need to firstly introduce a definition.

Definition 8. *Suppose* $w^1(t), ..., w^p(t)$, $p \in N$, $t \in [0, \infty)$ *are* p *functions that linearly independent, and satisfy* (L). *Then there is a smallest number* $\mu \in N$ *such that* $\lim_{t\to\infty} \wedge_{s=1}^{\mu} z^s(t) \to 0$, $(z^s \in span[w^1(t), ..., w^p(t)],\ s = 1, ..., \mu)$, *called* $\mu - number$ *of the function space* $w^1(t), ..., w^p(t)$.

Proposition 8. $\lim_{t\to\infty} x(t) \wedge y(t) = 0$ *for functions some* x *and some* y *that are all bounded. Denote them as* $x^1, ..., x^p$ *and* $y^1, ..., y^q$. *Suppose* μ_1, μ_2, μ_0 *be the* $\mu - numbers$ *of* $[x^1, ..., x^p]$, $[y^1, ..., y^q]$ *and* $[x^1, ..., x^p, y^1, ..., y^q]$. *Then we have the equation*

$$\mu_0 = \min\{\mu_1, \mu_2\}.$$

Remark 3. *With Proposition8 we can examine solutions to homogeneous differential equations, And convert the limit of wedges to asymptotically stability of other homogeneous equations.*

4 Nonhomogeneous linear equations

In this section, we will generalize Theorm 5 by generalizing their coefficient matrices.

$$\dot{x}^i = A^i(t)x^i + f^i(t), \quad i = 1, ..., k$$

and suppose $x^1, x^2, ..., x^k$ are their solutions respectively. Without loss of generality, we assume that $x^1, x^2, ..., x^k$ are linearly independent. And we assume that they are independent at each time t, which is a stronger condition.

we try to find an $n \times n$ matrix $C(t)$ such that $(A^i(t) - C(t))x^i(t) = 0$ for $i = 1, ..., k$.

$$\left(\begin{pmatrix} a_{11}^i & \dots & a_{1n}^i \\ \vdots & & \vdots \\ a_{n1}^i & \dots & a_{nn}^i \end{pmatrix} - \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix}\right)\begin{pmatrix} x_1^i \\ \vdots \\ x_n^i \end{pmatrix} = 0$$

$$\begin{cases} (a_{11}^i(t) - c_{11})x_1^i + \cdots + (a_{1n}^i - c_{1n})x_n^i = 0 \\ \vdots \\ (a_{n1}^i(t) - c_{n1})x_1^i + \cdots + (a_{nn}^i - c_{nn})x_n^i = 0 \end{cases}$$

We first consider the first row of the equation, and we have

$$\begin{pmatrix} x_1^1 & \dots & x_n^1 \\ \vdots & & \vdots \\ x_1^k & \dots & x_n^k \end{pmatrix}\begin{pmatrix} c_{11} \\ \vdots \\ c_{1n} \end{pmatrix} = \begin{pmatrix} a_{11}^1 x_1^1 + \cdots + a_{n1}^1 x_n^1 \\ \vdots \\ a_{11}^k x_1^k + \cdots + a_{n1}^k x_n^k \end{pmatrix}$$

And therefore we can choose $c_{11}, ..., c_{1n}$ such that the equation holds. Symmetrically we can choose the rest $c_{s1}, ..., c_{sn}$, $s = 2, ..., n$, therefore $C(t)$. Now we have

$$\dot{x}^i = C(t)x^i + f^i(t), \quad i = 1, ..., k$$

then we turnt it into the previous form.

Theorem 7. *Let* $x^i(t)$ *be a solution of*

$$\dot{x} = A^i(t)x + f^i(t),$$

where $f_i \in L_1[0,\infty)$, $i = 1, ..., k$. *Suppose that the homogeneous equation*

$$\dot{y} = A^i(t)y$$

is uniformly stable for all $i = 1, ..., k$ *and that we can choose* $C(t)$ *such that the k-th compound equation*

$$\dot{z} = C^{[k]}(t)z$$

is uniformly stable and asymptotically stable, Then there exist constants $d_1, ..., d_k$*, not all zero,such that*

$$\lim_{t\to\infty}(d_1x_1(t) + \cdots + d_kx_k(t)) = 0,$$

Remark 4. *We assumed that all the solutions are linearly independent at every time* t *to ensure that the corresponding matrix is always invertible. Without this assumption the determinant of the matrix may be* 0 *at some time. We think maybe those conditions can be solved and described with the language of* ***Measure Theory*** *and* ***Probability Theory****, which is currently beyond our ability.*